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Ecological Modelling 132 (2000) 33–50

ECOLOGICAL MODELLING

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# Fractal geometry applications in description and analysis of patch patterns and patch dynamics

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**Abstract**

Fractal geometry applications have recently been paid great attention in ecology. In this paper, I summarize the state of the art and introduce several updated developments in analysis and description of patch patterns and patch dynamics by means of Mandelbrot's fractal analysis, with an emphasis on my current research results and a personal view. These topics include geometric fractals, statistical fractals, information fractals, the fluctuation-tolerant fractals of dynamic patch size and shape, patch hierarchical scaling, fractal spatial patterns, multiple scale sampling and data analysis, fractal fragmentation of the landscape habitat into patches, fractal correlation in patchy systems, fractal cluster dynamics of vegetation systems, fractal mechanisms and ecological consequences, the spatio-temporal integrated approach and so on. The ecological significance of fractals in patch pattern and patch dynamics is discussed. A case study on fractal analysis of patch dynamics of southern Texas savanna landscape is given. Several limitations of fractal analysis in ecological applications are also addressed. © 2000 Elsevier Science B.V. All rights reserved.

*Keywords:* Fractals; Fragmentation; Landscape; Patch dynamics; Patchiness; Sampling; Spatio-temporal integration; Savanna; Scale; Vegetation system

## 1. Introduction

For about three decades, spatial analysis has been dominated by a style of model building which has sought high predictive understanding in numerical terms but has paid little attention to the geometry of spatial form. Mandelbrot's concept of a fractal, as one of these fast moving research fronts coupled with concepts of complexity, criticality, and self-organization, extends our usual ideas of classical geometry beyond those of point, line, circle and so on into the realm of the irregular, disjoint and singular. Fractals represent many kinds of patterns, including density, diversity, dendritic stream networks, geometrical shapes, mountainous terrain, and size distributions of islands (Mandelbrot, 1983). It has the potential to provide us with a new way to understand and analyze such natural spatial phenomena, which are not smooth, but rough and fragmented to self-similarity or self-affinity at all scales.

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0304-3800/00/$ - see front matter © 2000 Elsevier Science B.V. All rights reserved.
PII: S0304-3800(00)00303-3

Landscapes are spatially heterogeneous, and the structure, function, and change of landscapes are themselves scale-dependent (or scale covariance, although we normally use simple fractal description (or scale invariance) to present them in most landscape ecology literature). A landscape patch can be defined as a relatively homogeneous spatial cell or body (two and three dimensions) differing in appearance from its surrounding matrix (Wu et al., 1992). Heterogeneity of environmental resources, succession, and disturbance result in patches of diverse size, shape, type, and ecotone (or boundary) characteristics. The patch characteristics may be important factors in ecological diversity, stability, and function. The geometric features of heterogeneity, multiple scales and self-similarity/affinity are characteristic of a variety of patch spatial patterns in landscape. More generally, there are pansymmetries of biological spatial-temporal structure in nature, which are the pansystem holography in space and the pansystem repetition in time (Li, 1986). Recent studies have included measures of the fractal geometry of landscape and patch pattern analyses in ecological literature, such as, soil and landscape data analysis (Burrough, 1981, 1983a,b; Tyler and Wheatcraft, 1990; Bartoli et al., 1991; Young and Crawford, 1991), forest fire and cluster growth (MacKay and Jan, 1984; Vannimenus et al., 1985), root/tree and habitat structures (Williamson and Lawton, 1991; Zeide, 1991; Zeide and Gresham, 1991; Zeide and Pfeifer, 1991; Berntson, 1994; Haslett, 1994), spatial and landscape pattern analysis (Franklin and Forman, 1987; Gardner et al., 1987; Goodchild and Mark, 1987; Krummel et al., 1987; Milne, 1988, 1990, 1992; O'Neill et al., 1988; Palmer, 1988, 1992; DeCola, 1989; Wiens and Milne, 1989; Loehle, 1990; Rex and Malanson, 1990; Warner and Fry, 1990; Li, 1992; Li et al., 1992; Montgomery and Dietrich, 1992; Garcia-Moliner et al., 1993; Li, 1993; Loehle and Wein, 1994; Loehle and Li, 1996), microbial transport through heterogeneous porous media (Li et al., 1996), and ecological phase transitions (Li and Forsythe, 1992; Li, 1995a; Loehle et al., 1996).

In this paper, I review and evaluate these different fractal measurements in landscape ecology, including geometric fractals, temporal (dynamical) fractals, statistical (stochastic) fractals, information fractals, etc., try to build new models to describe dynamic patch shape and size by using fluctuation-tolerant fractals, especially in marine environments, and fragmentation of the habitat into patches by means of number-size distribution fractals, provide some methods to deal with different scale sampling data and fractal correlation in patchy systems, propose the spatio-temporal integrated methodology, and discuss mechanisms and ecological significance of fractals in patch pattern and patch dynamics. Fractal analyses of patch dynamics in southern Texas savanna landscape and a cellular automata model of spatio-temporal response of vegetation to disturbance are included. Several limitations and estimation problems of fractals in ecological applications are also addressed.

## 2. What are fractals?

Fractals are conceptual objects showing structures at all spatial scales, with a scale-dependent self-similarity (Mandelbrot, 1983, 1989; Barnsley, 1988). The shape of fractals is nonrectifiable, consisting of an infinite sequence of clusters within clusters or waves within waves. In rectifiable objects, increasingly accurate measurements based upon successive scale reductions give series converging to a limit: the true extent of the object. By contrast, in fractals the same procedure generates infinite series, according to the relationship $N(\varepsilon) \propto \varepsilon^{-D}$, where $N(\varepsilon)$ is a number measure corresponding to the scale unit $\varepsilon$ and $D$ is the fractal dimension. The length of the object is then $L(\varepsilon) \propto \varepsilon^{1-D}$ and $D > 1$. The length diverges as $\varepsilon \to 0$. In a volume of Euclidean dimension $E$ the volume occupied by an object of fractal dimension $D$ is given by $V(\varepsilon) \propto \varepsilon^{E-D}$. This parameter exceeds the topological dimension $d$ of the object and is generally not an integer, but less than the space dimension of the object, that is, $d < D < d+1$. For example, the fractal dimension of Koch's snowflake is $D = \log 4/\log 3 = 1.2618$.

Taylor (1986) suggested that a set should be called a fractal if these different computations all lead to the same value for the index, which we then call the dimension of the set.

Fractals are characterized by so-called 'symmetries' or pansymmetries (Li, 1986; Li et al., 1992), which are invariance under dilations and/or contractions. Hence, the best fractals are those that exhibit the maximum invariance. A fractal invariant under ordinary geometric similarity is called self-similar (Mandelbrot, 1983). 'Self-similar' has two meanings. One can understand 'similar' as a loose everyday synonym of 'analogous'. But there is also the strict textbook sense of 'contracting similarity'. It expresses that each part is a linear geometric reduction of the whole, with the same reduction ratios in all directions. Self-similar processes are invariant in distribution under judicious scaling of time and space, which are connected with the so-called 'renormalization group theory', 'critical phenomena', '$1/f$ noises', etc. Mathematically, the scaling coefficient or index of self-similarity is a non-negative number denoted $H$ (which is the first letter of the British Harold Edwin Hurst's last name); a process $X = \{X(t), t \in \Re\}$ is self-similar with index H if, for any $a > 0$, the finite-dimensional distributions of $\{X(at), t \in \Re\}$ are the same as those of $\{a^H X(t), t \in \Re\}$ (Samorodnitsky and Taqqu, 1994; Ch. 7). Self-similarity cannot be compatible with analyticity. Random fractals are self-similar only in statistical sense and to describe them it is more appropriate to use the term 'scale invariance' than self-similarity. By 'scale invariance' in ecology, we mean that scales are ecologically equivalent so that the same ecological conclusions may be drawn from any scale statistically. There are many different self-similar processes; however, most studies have considered those that have stationary increment. More recent developments have extended, in particular, to include self-affine, in that the reductions are still linear but the reduction ratios in different directions are different.

Fractals have been used to study nonlinear spatial and temporal phenomena (such as $D$ as a measure of complexity), but they can also be extended to abstract objects developing in a phase space, such as models of dynamic complex systems. Size–frequency distributions describing structured systems can also have a fractal dimension. There are several methods of measuring the fractal dimension which include changing coarse graining level, i.e. box-counting methods, information fractals, etc., using the fractal measure relations, i.e. perimeter/area/volume methods, using the correlation function, i.e. autocorrelations, semivariograms, etc., using the distribution function, i.e. hyperbolic distribution, and using the power spectrum, i.e. Fourier transformation, filters, wavelets, etc. (Li et al., 1992). Although the theoretical origins of fractals in measure theory may seem abstruse, the basic ideas of fractal geometry are extremely simple and intuitive, and one can begin to work with them very quickly.

Fractal dimensions can be positive, negative (Mandelbrot, 1990), complex (Pietronero and Tosatti, 1986), fuzzy (Feng et al., 1991), and multifractals (Mandelbrot, 1989), etc.

Generally, there are three properties of fractal forms: heterogeneity, self-similarity, and the absence of a characteristic scale of length. These geometric features are also characteristic of patch patterns in landscape. The fractal dimension $D$ has been shown to be a useful way to characterize the geometric structure of a number of these patchy spatial patterns (Milne, 1988).

The fractal concept is also useful for characterizing certain aspects of patch dynamics. Consider a complex process of landscape patch change that cannot be expressed in terms of a simple characteristic rate, but instead is regulated by a self-similar or self-affine mechanism in time. The multiplicity in time scales will be reflected in a power spectrum with a broad profile of responses. The fractal scaling between variations on different time scales will lead to a frequency spectrum having an inverse power-law distribution. A fractal analysis of a cellular automata model of spatio-temporal response of vegetation to disturbance in the later section is an example.

## 3. Current applications in landscape ecology

Recently, Milne (1990) and Sugihara and May (1990) reviewed fractal applications in ecological research and landscapes. Here I will select some interesting aspects associated with spatial patterns and patch dynamics, combine my current research in this field, and introduce them.

### 3.1. Perimeter versus area relationships

Krummel et al. (1987) used fractal models to show that patch shape varies with patch size. The relationship is generalized to fractal patches by the equality between area and patch length,

$$A = \beta L^{D}$$

where $A$ is patch area, $L$, patch perimeter, $\beta$, constant, and $D$, fractal dimension. By using data of patch area and patch perimeter, and relationships of $\log A = \log \beta + D \log L$, we can easily estimate the fractal dimension $D$. They found a marked ($P < 0.001$) discontinuity or scale breaking in $D$, with $D = 1.20 \pm 0.02$ at small scales and $D = 1.52 \pm 0.02$ at large scales. The discontinuity occurred at areas around 60–70 ha. Their result is interpreted to indicate that human disturbances predominate at small scales making for smoother geometry and lower $D$, while natural processes (e.g. geology, distributions of soil types, etc.) continue to predominate at larger scales.

In general, the area of fractal patches can be expressed as a function of perimeter raised to an exponent which provides the information of complexity change of average patch shapes; however, such fractal measurement cannot be used to characterize the nature of space-filling of ecological objects such as forest expansion (Loehle et al., 1996). Similar studies can be found in DeCola (1989), Rex and Malanson (1990), Haslett (1994), and Cheng (1995).

### 3.2. Information fractals of patch diversity

The information fractal dimension, $D_I$, is a generalization of the box-counting method that takes into account the relative probability of the patchy types which cover the landscape. Such dimension is related to the scaling relationship between the information needed to specify the position of a point and the accuracy to which the position is known. Therefore, the information fractal dimension as the natural measure dimension has been used in calculating the dimensionality of strange attractors (Farmer et al., 1983; Grassberger and Procaccia, 1983; Ruelle, 1989).

The information fractal dimension is given by

$$D_I = \lim_{\varepsilon \to 0} \frac{H(\varepsilon)}{\ln(1/\varepsilon)}$$

where $H(\varepsilon)$ is a given Shannon function (Shannon, 1948),

$$H(\varepsilon) = -\sum p_i \ln p_i$$

where $p_i$ is the probability of observing the $i$th patch element measured using samples of $\varepsilon$ units in size. The information fractal dimension of a measure corresponds to the dimension of the part of the measure that contains an arbitrarily large portion of the total measure as the cut-off $\alpha \to 0$. For very complicated landscapes, e.g. $n$-phase mosaics of different vegetation types in a natural landscape, we can extend the above formula to a generalized hierarchical diversity function, for example, Pielou's hierarchical diversity index (Pielou, 1977), to count the total patch diversity in the landscape. In practical applications, we could use the following relationship to estimate $D_I$, that is, the diversity $H(\varepsilon)$ will vary $\varepsilon$ according to

$$H(\varepsilon) = H(0) - D_I \ln \varepsilon$$

where $D_I$ as the lower bound to the Hausdorff Besicovich dimension or information fractal dimension. In many cases, the lower bound is numerically identical to it, e.g. linear regression estimation. Comparing with the fractal dimension $D$ from the box-counting method, in general, $D \geq D_I$ (Farmer et al., 1983). Only if all boxes have equal probability of the patchy types, $D = D_I$.

In our research about fractals in 'point' patch patterns, we found that information fractal dimensions varied with the different 'degree of randomness' parameters ($R$) of Clark and Evans (1954) (Li and Wu, unpublished manuscript). For instance, when $R > 1$ indicates regularity, we have fractals, $1.92 \leq D_I \leq 2$, where we used point patterns based on Wu et al. (1987). Fig. 1 shows different information fractal dimensions for a regular point pattern, random point pattern, random clumped point pattern, and aggregated clumped point pattern. Because a fractal dimension is scale invariant, it provides us with a new index to measure 'point' patch patterns and diversity. Information fractal dimensions also can be used in quantifying landscape habitat diversity (Loehle and Wein, 1994) and non-geometric ecological properties such as permeability (Loehle and Li, 1996) and uncertainty in ecological systems (Li, in preparation).

### 3.3. Patch hierarchical scaling

Different observational scales capture different aspects of structure, and these transitions are signaled by shifts in the apparent dimension of an object. This latter fact suggests an interesting application of fractals as a method for distinguishing hierarchical size scales of patches in nature, such as how to determine boundaries between hierarchical levels and how to determine the scaling rules for extrapolating within each level.

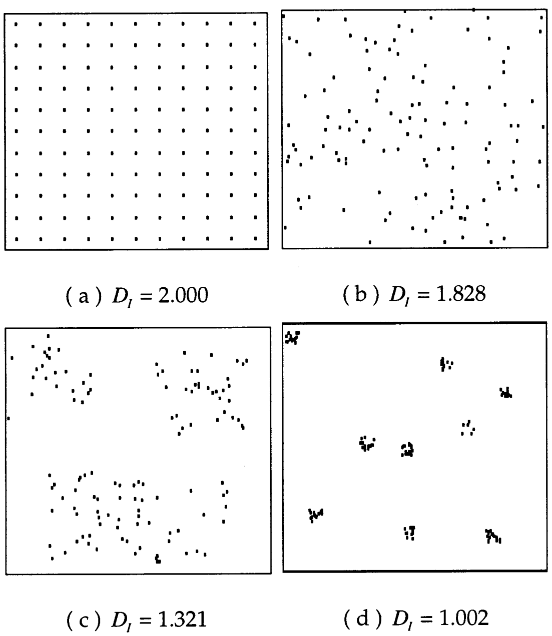


Fig. 1. Information fractal dimensions ($D_I$) in different point patterns. (a) Regular point pattern; (b) Random point pattern; (c) Random clumped point pattern; and (d) Aggregated clumped point pattern.

Bradbury et al. (1984) examined the possibility of hierarchical scaling in an Australian coral reef. They used the dividers method in transects across the reef to determine whether $D$ depends on the range of length scales. They found that three ranges of scale correspond nicely with the scales of three major reef structures: 10 cm corresponds to the size of anatomical features within individual coral colonies; 20–200 cm corresponds to the size range of whole adult living colonies; and 5–10 m is the size range of major geomorphologic structures. This showed that the shifts in $D$ at different scales appear to signal where the breakpoints occur in the hierarchical organization of reefs.

In our recent study on simple patch patterns, change of fractal dimensions seems to predict hierarchical scales of patch size and structure in nature, both of different mean grain densities within patch and spatial patterns between patches (Li, personal observation). Since the change in the fractal dimension may tell us something about the underlying physical and biological processes, $D$ could be used as a scaling indicator of patch phase transition, and may help us decide the appropriate scale of ecological experiment and management. The concept of discrete-scale invariance (i.e. the log-periodic correction to scaling) (Sornette et al., 1996) may also provide a new way to study patch hierarchical scaling relationships in nature.

Recently, we also found scale-break property in a semi-arid biome transition zone located at central New Mexico, USA, by using a multifractal approach to our Sevilleta vegetation transect data analyses (Over and Li, in preparation).

### 3.4. Fractal spatial patterns and modified Brownian dynamics

There are simple relationships between persistence measured by the parameter $H$ in modified Brownian diffusion models, and fractal exponents, $D$. Hastings et al. (1982) and Mandelbrot (1983) have discussed how fractal exponents may be incorporated into diffusion processes, as a scaling factor for normalizing increments in space and time. They find that $D$ may be used as an index of succession in circumstances where simple patch-extinction models are reasonable. Sugihara and May (1990) consider that it would be interesting to follow up these provocative anecdotes with careful studies to determine to what extent $D$ computed from snapshots can be used as an index of physiological state or persistence of patches in time, and how such persistence may relate to the spatial scales involved. Recent studies have shown that many of nature's seemingly patch shapes can be effectively characterized and modeled as random fractals based upon generalizations of fractional Brownian motion (Voss, 1988). Extending the ideas in fractal correlation analysis of patchy systems will be addressed later.

### 3.5. Others

There are other fractal applications which I will not expand upon. These include number/diameter relationships, mass relations, habitat space and home range (Loehle, 1990), geometry of plankton patches in turbulent flows (Frontier, 1987), mixed fractal models, percolation theory (MacKay and Jan, 1984; Grimmett, 1989; Loehle et al., 1996), a space–time multifractal theory (Over and Gupta, 1996), etc.

## 4. Multiple scale sampling and data analysis

Biotic and metric scale-dependent structures exist in landscapes. Biotic scale dependence originates from the different responses of organisms to the abundance of a resource. Metric scale dependence results when physical processes produce statistically similar aggregations of abiotic quantities such as water, temperature, or minerals (Milne, 1990). From a fractal point of view (Mandelbrot, 1983, 1989), we consider a sampling space dimension $D_s$ and a specific ecological phenomena dimension $D_p$ embedded in a space of dimension $E$. If $D_s + D_p > E$, we will obtain a nonzero measure, that is, a specific sampling space dimension $D_s$ used in space of dimension $E$ can only detect such phenomena of different dimension $D_p > E - D_s$. Lovejoy et al. (1986) augured that the characteristics of measuring network inhomogeneity by the fractal dimension raises new problems concerning the detectability of sparse phenomena. They suggested a new criterion for evaluating measuring networks: to detect geophysical phenomena, not only must a network have sufficient spatial resolution it must also have sufficient dimensional resolution. Because our sampling space is at multiple scales, of course, such sampling is multiple scale sampling.

In general, the sum of random variables with a common distribution becomes a random variable with a distribution of different form. Data from different scales have their corresponding probability distributions. However, statistical fractals show us that the sum of a large number of identically distributed random variables has self-similarity with distribution as each of contributors to the sum. The class of distributions having this property is called the Levy distribution $p(X; \alpha, \theta) \propto X^{-\alpha-1}$ with $\alpha \neq 2$ (Mandelbrot, 1983; Samorodnitsky and Taqqu, 1994). An appropriate linear transformation makes the sum of random variables obey the same distribution. That implies that when we blend data from different sources, we first need to find a suitable transformation to form a stable distribution, although there are some statistical methods to deal with data from unequal probability sampling and different scales, e.g. the weighted distribution method (Patil and Taillie, 1989).

From this idea, the author obtained a practical computation method for calculating the mean and the variance of data from two different scale samples (Li et al., 1992). We define weight mean and variance as

$$w = \frac{Var(1)}{Var(1) + Var(2)}$$

$$X = wx_1 + (1 - w)x_2$$

$$Var = w^2 Var(1) + (1 - w)^2 Var(2)$$

where $w$ is the weighted factor; $Var(1)$, $Var(2)$, variances from data of scale #1 and #2; $x_1$, $x_2$, means from data of scale #1 and #2; $X$, $Var$, the mean and the variance from blending two scale data.

For example: data from Rougharden (1977) are as follows.

…100, 150, 150, 100, 50, 50, 100, 150, 150, 100, 50, 50, 100,… (1)

MEAN(1) = 100, *Var*(1) = 1667, Patch length = 3;

…100, 125, 125, 125, 125, 125, 100, 75, 75, 75, 75, 75, 100,… (2)

MEAN(2) = 100, *Var*(2) = 521, Patch length = 6.

If we blend two sequences (having different characteristic patch scales) together to calculate their mean and variance, MEAN = 100, and *Var* = 997 (not 2188 in Rougharden (1977)).

It has been reported that the mixture distributions or contaminated normal model (usually these distribution densities have the same mean but different S.D.) can lead to a stable Paretian or Levy distribution (e.g. Hsu et al., 1974). In such models, the underlying 'categories' do not necessarily have a direct ecological interpretation (Titterington et al., 1985). We need to be careful to deal with such models for multi-scale data analysis, and should not jump to a conclusion or interpretation, which is considered ecologically to be significant too soon.

## 5. Fractal fragmentation of habitat into patches

The fragmentation of the habitat into discrete patches is a topic of concern in relation to biodiversity conservation and resource management efforts (Harris, 1984; Pimm and Gilpin, 1989). It is an important feature of landscapes. Landscape habitat is fragmented by joints of natural processes and human and natural disturbances. Fragmentation is thought to play a dominant role in determining the size–frequency relationship for astrophysics (Brown et al., 1983). There are a variety of ways to represent the size–frequency distribution of habitat fragments or patches. In the following, I will use a simple power law relation to define the fractal distribution for quantifying the habitat fragmentation processes and give a hypothesis about the fragility or vulnerability of landscape habitat based on the result of the Turcotte's renormalization group approach to the problem of fragmentation (Turcotte, 1986).

In fractal analysis, the definition of a fractal could be given by the relationship between number and size. The number $N(\varepsilon)$ of objects with a characteristic linear dimension greater than $\varepsilon$ is given by

$$N(\varepsilon) \propto \varepsilon^{-D}$$

where $D$ is fractal dimension. In many cases, the frequency–size distribution of fragments or habitat patches over a wide range of scales satisfies a fractal distribution. Fragmentation is often a scale invariant process (Turcotte, 1986). We can directly use the above formula to estimate the habitat patch fractals. An alternative power-law distribution can be written

$$N(m) = Cm^{-b}$$

where $N(m)$ is the number of habitat patches with a biomass greater than $m$, $C$ is a constant, and $b$ is a scaling exponent. Noting that $m \propto \varepsilon^{-3}$, we can find from a comparison with the number-size fractal distribution that

$$D = 3b$$

The power distribution is mathematically equivalent to a fractal distribution. Similarly, the power-law biomass distribution is given by

$$\frac{M(\varepsilon)}{M_{\mathrm{T}}} = \left(\frac{\varepsilon}{\sigma}\right)^{\alpha} \text{ with } D = 3 - \alpha$$

where $M(\varepsilon)$ is the cumulative biomass of habitat patches with a radius (volume$^{1/3}$) less than $\varepsilon$, $M_{\mathrm{T}}$ is the total biomass, and $\sigma$ is related to the biomass size. We can also write a power-law distribution in terms of a number density function

$$f(m) = Am^{-s} \text{ with } D = 3(s-1)$$

where $f(m)\,\mathrm{d}m$ is the number of habitat patches in the biomass range between $m$ and $m + \mathrm{d}m$, and $A$ is a constant. Both of the distributions satisfy the fractal condition. They can be used as an indicator of the fragmentation of the landscape habitat into patches and landscape change.

Using the renormalization group approach to scale invariant problems with fractal distributions, Turcotte (1986) proposed two renormalization group models to the fragmentation problem. The models yielded a fractal behavior for fragmentation but gave different values for the fractal dimension. Turcotte indicated that the fractal dimension is a measure of the fracture resistance of the material relative to the process causing fragmentation. His conclusion can be used as our hypothesis about the fragility or vulnerability of habitat fragmentation. That is, a more fragile or vulnerable landscape habitat may be associated with a smaller fractal dimension. The result of Krummel et al. (1987) supports my hypothesis. I can re-interpret their result to imply that landscape habitat response to human disturbances is more fragile or vulnerable than to natural processes. It may be useful for managing and conserving the ecological habitat.

Several other studies in fractal measures of habitat fragmentation to quantify human impact and describe various habitat structures or patchy landscape features can be found in Milne (1988), Palmer (1988, 1992), Williamson and Lawton (1991), Haslett (1994), and Loehle and Wein (1994).

## 6. Fractal correlations in patchy systems

Ecologists have recognized the effects of spatial heterogeneity on ecological systems (Pickett and White, 1985; Franklin and Forman, 1987; Milne, 1990, 1992; Kolasa and Pickett, 1991; Li and Forsythe, 1992; Palmer, 1992; Garcia-Moliner et al., 1993; Li, 1993; Li and Archer, 1997). There are several ways to analyze and describe spatial heterogeneity of patchy systems. Spatial statistics or geostatistics are necessary to document spatial relationships, e.g. spatial correlation. For spatial heterogeneity study, however, current geostatistical methods have severe limitations (Li and Loehle, 1995). For many types of nonstationary, singular, and discontinuous patchy data, geostatistical methods are not capable of accounting these features of patchy landscape and do not allow one to detect or describe multi-scale structures of patchy systems. They also do little to reveal the dynamics of changes in patch patterns over time. Thus, a new method is needed to describe heterogeneous patch structures and patterns.

In reality, most landscapes exhibit patterns intermediate between complete spatial independence and complete spatial dependence. The fractal dimension $D$ provides a measure of the degree of correlation between patches over space or time. The correlation between nearest neighbors in a spatially heterogeneous system (the three-dimensional profiles of patch in the ocean) is defined by the fractal dimension describing the degree of heterogeneity in a size-independent fashion. By using the classical definition of the correlation coefficient, the result was that the nearest neighbor correlation with respect to local property was expressible directly by the correlation coefficient, $r_1$

$$r_1 = 2^{3-2D} - 1 = 2^{2H-1} - 1; \quad 0 \le H < 1$$

where the Hurst coefficient $H = 2 - D$ for one-dimensional signals such as vegetation line transect data.

If there is no spatial correlation, $r_1 = 0$, then the local patch patterns are completely random and the fractal $D = 1.5$, or the Hurst coefficient $H = 0.5$ (Voss, 1988). With perfect correlation, $r_1 = 1.0$, and the patches are uniform everywhere. If we have fractal dimension $D = 1.2$ of the patch system, correlation coefficients between adjacent patches are 0.516.

Similarly, we can define the correlation coefficient between patches centered two units apart:

$$r_2 = \frac{3^{2H}}{2} - \frac{3}{2} - 2r_1 = \frac{3^{2H}}{2} - 2^{2H} + \frac{1}{2}$$

This equation can be extended for aggregates of $n$ adjacent patches to give

$$r_n = \frac{1}{2}[(n-1)^{2H} - 2n^{2H} + (n+1)^{2H}]$$

which is valid for $n \ge 1$. With increasing $n$, the correlation initially falls off rapidly, but at large $n$ the diminution is slow. In fractal patch systems with $H < 0.7$, the values of $r_n$ for $n > 1$ are so low that we can consider that patchy systems are regarded as noise.

From the fractal point of view, a scaling form can be written with an arbitrary scale factor $b$:

$$P([X - X_0], |t - t_0|) = b^H P(b^{-H}[X - X_0], b|t - t_0|)$$

and expresses a relation between the time domain and the space domain (Feder, 1991). This equation may provide a promising way to view fractal space–time structure. Feder further derived a similar equation of correlations between past and future increments as we did above for the nearest neighbor correlation.

We can also use geostatistical tools for characterizing fractal correlations of patchy systems. The semivariance $\gamma(h)$ and its relationship with the Hurst exponent are defined as

$$\gamma(h) = \frac{1}{2} E\{[x(t+h) - x(t)]^2\} \propto h^{2H}, \ 0 \le H < 1$$

where $E$ denotes statistical expectation, so that $2\gamma(h)$ is the expected squared difference between values separated by a time or space lag $h$. An estimate of $\gamma(h)$ is obtained using the sample semivariance $s^2(h)$, defined by

$$s^2(h) = \frac{1}{2(n-h)} \sum_{i=1}^{n-h} [x(t+h) - x(t)]^2$$

Theoretical linkage between multifractal models and spatial statistics with geological application has been investigated by Cheng and Agterberg (1996). We can also use spectral analysis to obtain fractal dimensions of patchy systems. For example, we can use $S(f) \propto f^{-\beta}$, where $\beta = 2H + 1$ with $0 < H < 1$, $1 < \beta < 3$ and $D = 2 - H$. However, we have to be careful on fractal estimation via power spectra and variograms; there are reports on biased estimates of the $D$ value in the literature (Wen and Sinding-Larsen, 1997).

## 7. Fluctuation-tolerance fractals in dynamic patch size and shape

Levin and Paine (1974, 1975) and Paine and Levin (1981) have developed a mathematical model for the spatial and temporal patterns of patch dynamics. However, we try to use fractal theory in dynamic patch size and shape, especially in marine environments. From the fractal viewpoint, change of patch size and shape in space has a multiplicity of spatial–temporal scales and multifractal dissipation. My approach is assuming having self-similarity of patch change and ability of maintaining patch integrity while allowing for a broad spectrum of variations both in space and time. According to some results from fluctuation-tolerance fractals in complex physiological structure and processes (West, 1990), we similarly obtain the probability distribution of a scale of patch size presented in time $t$ which could be related to the asymptotic statistics of a Levy process, and patch size–frequency distributions. Because the Levy index can be related to the fractal dimension of the underlying process, many studies have shown where there exist some relationships between generalized diffusion equations and fractal random walks (Mandelbrot, 1983). In a fractal stochastic process, not only does the process itself display a kind of self-similarity, but also the distribution function characterizes the statistics of the process. If $X(t)$ is a fractal random patch dynamic function, then for constant $\beta > 1$ and $\alpha > 1$, we have $X(t) = \beta^{-\alpha} X(\beta t)$. For example, a given realization $X(t)$ is identical with one that was stretched in time by $\beta t$ and scaled in amplitude by $\beta^{-\alpha}$, where $\alpha$ is related to the fractal dimension. Subject to certain ecological assumptions we can obtain the exponents $\phi$ for the decay of the power spectrum ($S(f) \propto f^{-\phi}$), $\tau$ for patch-size distributions ($D(s) \propto s^{1-\tau}$), and fractal dimension $D_f$ of patch ($s \propto L^{D_f}$), by relating them to the spatial anisotropic exponent $\zeta$, the usual dynamical scaling exponent $z$, and the anomalous spatial correlation exponent $\chi$ (Li, personal observation).

In addition, some studies show that diffusion-limited aggregation (DLA) is the physical origin of fractals (Pietronero, 1989). Perhaps these theories could provide us with a possible explanation about formation mechanisms, and ecological and evolutionary consequences of patch patterns and patch dynamics (Li et al., 1992). I will talk about this issue further in the next section.

## 8. Fractal mechanisms and ecological consequences

A realization of the importance of fractals in describing a large variety of patterns occurring in nature across different spatial–temporal scales has been one of the major developments of the last decade. The key problem is to understand why nature gives rise to fractal structures or power-law distributions (PLDs). Do we simply have to accept their existence as 'God-given' without further explanation or is it possible to construct a new dynamical theory of the ecology of fractals and PLDs? Recent studies in physics and other fields could provide us a new insight to analyze and explain the pattern in terms of the ecological processes believed to underlie them (Li, 1992, 1993, 1995a; Li and Forsythe, 1992; Li et al., 1992).

We concern the behavior of spatially extended dynamical ecological systems, that is, systems with both temporal and spatial degrees of freedom. However, there is a lot to learn on the spatio-temporal evolution of these complex systems; Actually, it is hard to believe that long-range spatial and temporal correlations can exist independently. A local pattern cannot be 'robust' and remain coherent over the long term in the presence of any amount of 'noise', unless stabilized by the interactions with its environment. And a large, coherent spatial structure cannot disappear (or be created) instantly.

Bak and coworkers (Bak et al., 1987; Bak and Chen, 1991; Bak, 1996) have suggested that the large temporal fluctuations, and the spatial self-similarity are two sides of the same coin: 'Self-Organized Criticality (SOC)'. The idea is that the systems operate persistently out of equilibrium at or near a threshold of instability. The systems evolve automatically to this critical state without any fine-tuning of external fields. Hence the criticality is self-organized. The idea has been confirmed theoretically and numerically for a sandpile model of avalanches that the critical point is stable in the renormalized macroscopic limit.

Li (1992, 1995a) and Li et al. (1992) suggested a possible explanation about formation mechanism, ecological and evolutionary consequences of fractal patch patterns and dynamics by using SOC concept. On the basis of this concept which appears in a wide class of dissipative dynamic systems with spatial degrees of freedom (Bak and Chen, 1991), we could approach criticality of patch dynamics since certain extended dissipative dynamical systems naturally evolve into a critical state, with no characteristic time and space scales (Li and Forsythe, 1992).

We used a cellular automata-based simulation model of a multi-patch landscape subjected to different disturbances (i.e. intensity or extent and scale or grain of disturbance) to study criticality of spatially heterogeneous vegetation landscape response to disturbances (Li and Forsythe, 1992; some results also reported in Li and Archer (1997)). Our results (Figs. 2 and 3) indicate that disturbance-influenced vegetation systems will exhibit SOC states and that ecosystems may operate persistently out of equilibrium at or near a threshold of instability and coevolve to the edge of chaos. These results are in accord with the SOC hypothesis and other studies (e.g. Kauffman and Johnsen, 1991; Ito and Gunji, 1994). Following the work by Rinaldo et al. (1993), I analyzed the structure of natural landscapes and their dynamics of evolution via fractal and wavelet analyses (Li, 1995a). I argued that the principle of minimum energy expenditure and SOC concept are of crucial importance for understanding basic mechanisms that govern landscape dynamics and evolution.

However, after carefully examining those results, I have recognized that even these models do reproduce the fractal characteristics of landscapes or river networks, but finally lead to stationary patterns or distributions which is clearly inconsistent with the SOC concept that predicts a permanent fluctuation around a long-term equilibrium. Minimizing the total energy dissipation these models used is thermodynamically for near equilibrium, not for far-from-equilibrium systems. From this perspective, the model limitation is clear. The SOC concept basically depends on the long-term equilibrium between driving forces and dissipative. For a simple simulation like a sandpile model, this assumption is appropriate. When we use it for large-scale landscape dynamics such as biome transitions, the conditions cannot be expected to be constant. The behavior within the transition phase may be far outside the range suggested by the corresponding equilibrium states, so that the properties of these states cannot be used for estimating the behavior during the transi-

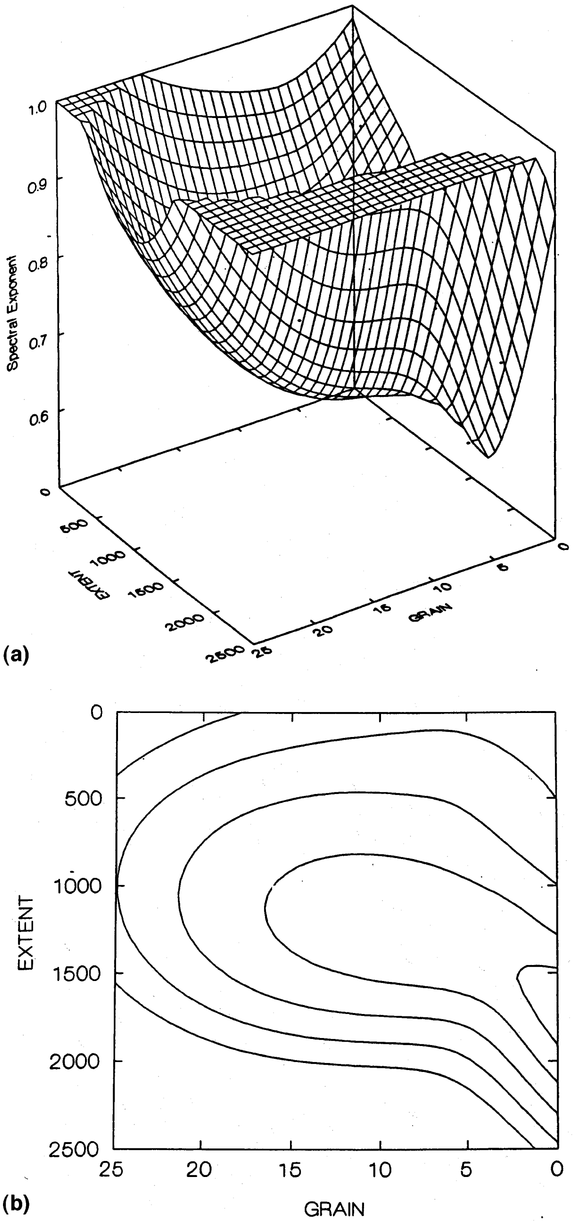


Fig. 2. Spectral analysis results of a cellular automata vegetation patch dynamic model: (a) 3-D plot of spectral exponents of vegetation landscape response to disturbances with different intensities ( = extent) and scales ( = grain); and (b) its corresponding contour plot.

tion phase. I also feel that the SOC concept is too simple and too vague for complex systems, and lack analytical support. There is no general agreement about ingredients necessary to create the self-organized critical state. Some SOC models are even not critical in the presence of local dissipa-

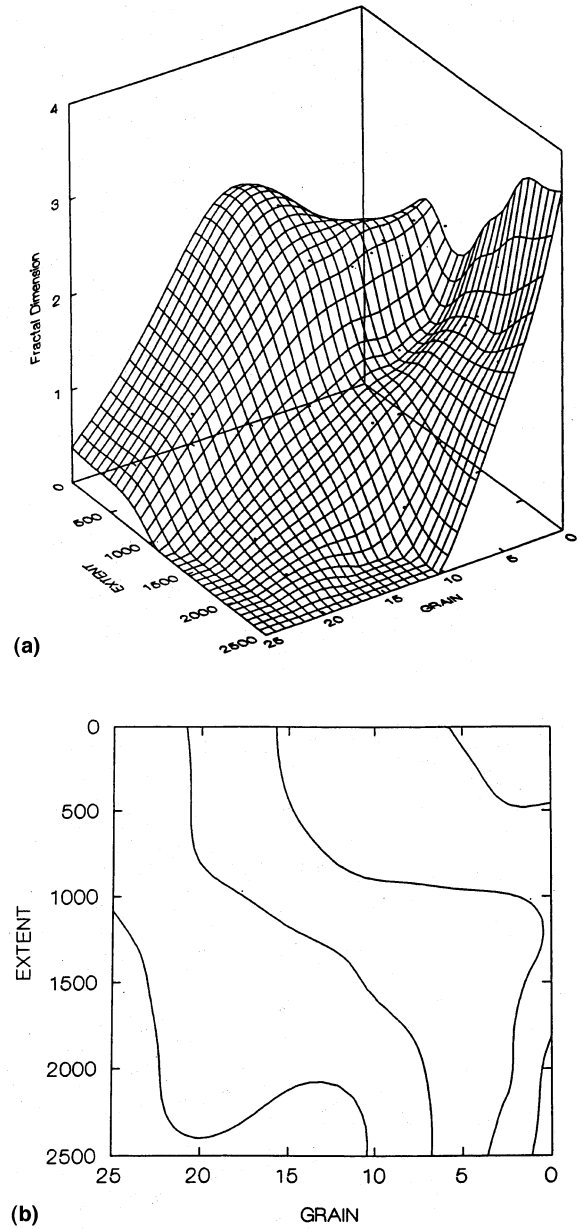


Fig. 3. Spatial size-distribution analysis results of a cellular automata vegetation patch dynamic model: (a) 3-D plot of fractal dimensions of vegetation landscape response to disturbances with different intensities ( = extent) and scales ( = grain); and (b) its corresponding contour plot.

tion. Therefore, before we talk about how nature works by using SOC framework, we may need to ask how SOC works first.

Mathematically, we can demonstrate that interaction of simple elements on the microscopic level may result in the emergence of complex scaling features of the system on the macroscopic level, and that internal and external noise/force, multiplicative stochastic processes, and purely mixing distributions can induce ecological scaling (Li, in preparation). Those results are based on very straightforward physical and biological principles and very limited assumptions, and reflect the co-evolutionary or adaptive processes of ecological entities interacting with their stochastic biotic and abiotic environment. Some of these results have very strong analytical support (e.g. Kesten, 1973; Sornette and Cont, 1997; Takayasu et al. 1997; Sornette, 1998), and could be served as much more general mechanism of producing fractal or power-laws.

The ecological significance of fractals is that they describe very compactly, the relation between the spatial variability of the patches, and the space–time scale. But we may also need to be careful about universality of fractals. If everything is expected to be universal, the term loses its depth: when someone insists that Nature is simple and that essentially everything is power-law (fractal) and universal, what they are actually telling you is that Nature is shallow.

## 9. A case study: fractal patch (cluster-phase) dynamics in southern Texas savanna landscape

Trends toward increased woody plant abundance in temperate and tropical grasslands and savanna in recent history have been reported world-wide. We have little knowledge of the rates, dynamics, patterns or successional processes involved. To determine the long-term patterns and dynamics of vegetation cluster-to-cluster (or patch-to-patch) interactions in southern Texas savanna landscape we defined different fractals and fractal relationships to describe (a) cluster growth relationships; (b) changes in the size and shape of clusters; (c) degree of coalescence or fragmentation; and (d) spatial pattern shifts of different types of vegetation clusters during succession in southern Texas subtropical savanna (Li, 1993; also Li and Archer, unpublished manuscript). Some of fractals and their relationships included area–perimeter fractals, cluster-size distribution fractals, self-affine fractals, correlation fractals, and information fractals that I already introduced before.

The Rio Grande Plains of southern Texas and northern Mexico offer some distinct examples of processes involved in the physiognomic conversion of grassland and savannas to woodlands. The potential natural vegetation of this region has been classified by plant geographers as *Prosopis*/*Acacia* savanna (e.g. a grassland with scattered woody plants). However, the present-day vegetation is a subtropical thorn woodland that occupies about 12 million ha in Texas alone. Rates, patterns and processes involved in succession from grassland to woodland have been the focus of investigations at the La Copita Research Area (Texas A&M University Research Station) in southern Texas since 1984. The process begins when the leguminous shrub, *Prosopis glandulosa* (mesquite) establishes in herbaceous zones. The seeds of *Prosopis* are widely dispersed by livestock and establish readily on sites where fire and competition from grasses have been reduced by grazing. As the *Prosopis* plant grows, it modifies soils and microclimate and facilitates the ingress and establishment of other shrubs. These clusters of woody vegetation which form around the *Prosopis* nucleus enlarge over time as new plants appear and existing plants grow. As the canopies of the woody clusters develop, light attenuation increases such that herbaceous production in well-developed clusters is only 20% of that in herbaceous zones. As new clusters are initiated and existing clusters expand, coalescence will eventually occur and a grassland will have become a woodland. For the detailed description, see Archer et al. (1988), Li (1995b), and Li and Archer (1997).

Estimating fractal dimensions were used in following techniques: (1) Fractal kinetics of aggregation processes: this approach is based on cluster–cluster aggregation in fractal growth (Vic-

Table 1
Local growing fractals of woody clusters at study sites from black and white aerial photographs in southern Texas

| Period | Fractal dimension | Roughness |
|---|---|---|
| 1941–1960 (DRY) | 1.0819 | 0.4603 |
| 1960–1983 (WET) | 1.1406 | 0.6328 |

Table 2
Anisotropic characteristics of correlated growth of woody clusters

| Year | Fractal dimension (roughness) in $X$-direction | Fractal dimension (roughness) in $Y$-direction |
|---|---|---|
| 1941 | 1.1478 (5.7231) | 1.2250 (5.2634) |
| DRY | | |
| 1960 | 1.1588 (5.6976) | 1.1189 (5.3163) |
| WET | | |
| 1983 | 1.1526 (5.7051) | 1.1189 (5.9614) |

Table 3
Changing complexity of the size and shape of all vegetation clusters

| Year | Climate condition | Fractal dimension |
|---|---|---|
| 1941 | | 1.8732 |
| | DRY | |
| 1960 | | 1.8337 |
| | WET | |
| 1983 | | 1.9377 |

Table 4
Degree of coalescence and fragmentation in southern Texas savanna landscape

| Vegetation type | Fractal dimension | Roughness |
|---|---|---|
| Herbaceous | 1.4061 | 0.6505 |
| Pioneer cluster | 1.3120 | 0.6096 |
| Mature cluster | 1.2908 | 0.5930 |
| Coalesced clusters | 1.2592 | 0.5889 |
| Woodland | 1.1189 | 0.5128 |

sek, 1989). According to Smoluchowski equation and their dynamic scaling, average patch size change can be described as

$$\frac{dS(t)}{dt} \propto [S(t)]^{\varsigma}$$

where $\varsigma$ is the dynamic scaling exponent related to fractal dimension. Based on black and white aerial photographs (1941–1983), we can calculate local growth fractals of woody patches in Table 1; (2) Correlation functions, self-affine fractal and spectral analysis (see Section 6): we can have fractal and roughness measure of anisotropic characteristics of correlated growth of woody patches in Table 2; (3) Area–perimeter relationships (see Section 3.1): we use area–perimeter fractal measure for characterizing the complexity of the size and shape of vegetation patches in Table 3; (4) Cluster-size distribution functions (see Sections 5 and 7): we use such distribution fractal measure for quantifying the degree of coalescence and fragmentation in the landscape (Table 4); (5) Information dimensions (see Section 3.2): we use information fractal measure to describe spatial pattern shifts of the landscape during succession. Tables 1–5 show that our method looks very favorable and is straightforward for understanding and identifying cluster-phase (patch) processes and successional mechanisms in vegetation systems.

## 10. Summary and discussion

Euclidean geometry is so familiar to us that we often forget that it is essentially hypothetical, particularly in its application to ecology and biology. For example, Euclidean analysis implies that the notions of length $\eta$, surface $\eta^2$ and volume $\eta^3$ have only a hypothetical meaning. The concept of fractal geometry provides us a new insight on analyzing and quantifying the spatial variability

Table 5
Spatial pattern shifts of southern Texas savanna landscape during succession

| Year | Fractal dimension | Roughness |
|---|---|---|
| 1941 | 1.7905 | 5.7130 |
| DRY | | |
| 1960 | 1.8282 | 5.5968 |
| WET | | |
| 1983 | 1.9130 | 6.2221 |

of patch patterns and patch dynamics. Patch systems operate over a broad range of spatial and temporal scales. Although such systems are often studied as discrete units of a landscape, in fact they are never completely isolated from each other. Rather, all ecological systems are more or less strongly linked to their neighbors through the transport of water, energy, organic matter, and mineral elements. In other words, one ecosystem's outputs are its neighbors' inputs. In order to understand and characterize ecological patterns and processes as well as their dynamics, we have to address scale and scaling issues. Scale interactions and cross-scale dynamics among patches or ecological systems become important. In general, we need to consider the following scales: (1) Temporal scale: (a) the lifetime/duration; (b) the period/cycle; and (c) the correlation length/integral scale; (2) Spatial scale: (a) spatial extent; (b) space period; and (c) the correlation length/integral scale; and (3) 'Organism' scale: (a) body size/mass; (b) species-specific growth rate; (c) species extinction rate; (d) the life span; (e) the home range; (f) niche, and so on. In practice, we have to identify: (1) Process scale: Only those processes which have comparable scales can significantly affect biogeochemical interactions and ecosystem dynamics and act as constraints on ecological systems. Scaling up from small to large cannot be a process of simple linear addition: nonlinear processes organize the shift from one range of scales to another. (2) Observation scale: (a) the spatial (temporal) extent/coverage of a data set; (b) the resolution/spacing between samples; and (c) measurement scale or the integration volume/time of a sample. Ideally, processes should be observed at the scale they occur. However, this is not always feasible. Processes larger than the coverage appear as trends in the data, whereas processes smaller than the resolution appear as noise. (3) Modeling/working/management scale: The modeling or working scales generally agreed upon within the scientific community are partially to processes and partially to the application of ecological models. Emerging ecological processes, observation methods and modeling together are a great challenge for ecological theory development and application. Our observations of the real ecosystems have finite resolution, and our computers have limited capabilities. Therefore in models we must divide the behavior of the system, which occurs on all scales, into the component that is explicitly resolved in the model and an unresolved, smaller-scale component. One key to the scale problem is to understand how the behavior of the system at different scales is accounted for in models. Because landscapes are moving targets, with multiple potential futures that are uncertain and unpredictable, ecosystems or landscape management has also to be flexible, adaptive, and experimental at scales compatible with the scales of critical ecosystem functions. Fractal analysis as a tool for addressing problems of scale and hierarchy allow ecologists to view landscape patch patterns and dynamics at multiple spatial and temporal scales and thereby achieve predictability in the face of complexity, and suggests that patch landscape properties will be a function of the scales of measurement and that traditional concepts of stationarity and averaging in stochastic approach may not capture the total nature of heterogeneity. In addition, statistical fractals offer a viable way to analyze discontinuous, inhomogeneous processes in natural systems.

Future directions in ecology will be strongly influenced by methodological advance, especially technologies imported from other disciplines (Wiens, 1992). Fractals will play a very important role in building a spatially explicit ecology because fractals not only have obviously advantage in describing the following three related contexts: geometric, temporal (dynamical), and statistical, but also fractals provide a bridge to concentrate chaos theory, fractal analysis, wavelet analysis (Gao and Li, 1993; Li and Loehle, 1995), scaling analysis and spectral analysis into a spatio-temporal integrated methodology. In order to enlarge ecological dynamics to fractal dynamics (or more generally speaking, to scale dynamics) in an ecological system, it is clear that the future applications of fractals and their underlying nonlinear space–time dynamics in a fractal space, together with high speed computation, will continue to bring ecological, physical and mathematical sciences together for work on real problems that were formerly thought to be outside some of the artificially set ranges in each field.

Fractals, like other mathematical models in ecology, have their limitation (e.g. Shenker (1994)), especially as it is a developing theory and method. For example, the fractional Brownian motions model the often observed power-law relations between the variance of a soil property and the length of transect sampled. However, this model fails to account for abrupt changes of the mean (i.e. soil boundaries), for second-order stationarity and for the non-self similarity of variations at different scales that are observed in real data (Burrough, 1983b). Multifractals could be used to account for such change or ecological phase transitions. For another example, there are severe theoretical difficulties in estimating the index, which stem from two different causes: (1) the dimension is local property defined in terms of behavior of the set in a box $B(x, \varepsilon)$ as $\varepsilon \to 0$, and we cannot let $\varepsilon$ go to zero in real data, the behavior over the observable range could be quite different from its limiting behavior; (2) It is impossible in practice to distinguish between the set $E$ and its closure $\bar{E}$. This means that a countable set $Q$ such that $\bar{Q}$ is thicker than the set $E$ that we want to observe might well dominate that calculation even though $Q$ is only countable. Theoretically, when the selected scale approaches zero and is larger than the threshold scale, the estimated fractal dimension $D'$ is the real fractal dimension $D$, otherwise, $D' < D$. Several recent studies addressed the problems of the fractal estimation (Cutler and Dawson, 1989; Lapsa, 1992; Reeve, 1992; Wen and Sinding-Larsen, 1997). A careful quantitative analysis of spatial patterns and patch dynamics using the concept of fractal geometry would be a valuable contribution that is, so far, lacking.

Self-similarity may be extreme restrictions in a particular situation (e.g. Simberloff et al., 1987). However, it is still possible and useful to apply the general idea to a natural system and define its fractal dimension. Although the concept of fractal dimension defined by Hausdorff may be applicable even to the sets that may not be self-similar, we have to recognize that fractal distributions may only be valid over a limited range. We have to address scale-break, scale covariance, and dynamic scaling carefully in our studies.

Fractal geometry was introduced into natural sciences less than three decades ago. In a short time, the fractal concept has turned from an esoteric mathematical idea into a useful tool in many branches of pure and applied science, including ecology (over 80 published papers with ecological applications are good evidence). The field has become mature and very sophisticated. It is my hope that this paper will stimulate work in this area and shed new light on the quantification of patch temporal and spatial variability and heterogeneity across spatial and temporal scales.

## Acknowledgements

This paper grew out of my presentations at several conferences including Patch Dynamics Workshop (Ithaca, NY) and ISEM'95 (Beijing, China) during 1991–1997. I am indebted to Simon Levin, Tom Powell and John Steele for their kind invitation and support, which resulted in my first draft of this manuscript in 1991. Steve Archer, Bill Forsythe, Craig Loehle, Sven Jørgensen (Editor-in-Chief), Tom Over, Ignacio Rodriguez-Iturbe, Wally Wu, and an anonymous referee provided helpful reviews and comments on the earlier versions of this manuscript. This work has been partially supported by the U.S. National Science Foundation (BSR-91-09240, DEB-93-06679 and DEB-94-11976), DOE/Sandia National Laboratories (BE-0229) and Argonne National Laboratory, Texas A&M University, and the University of New Mexico. This is Sevilleta LTER publication no. 156.

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